Introduction to Real Analysis I

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Chapter 1

The Real Number System

1.1 Review
1.2 Ordered Field Axioms
1.3 Completeness Axiom

Theorem 1.1 (Archimedean Principle). Given $a, b \in \mathbb{R}$ with $a > 0$, there is an integer $n \in \mathbb{N}$ such that $b < na$.

Proof. If $b < a$ then take $n = 1$. So we have $b < na$. If $a < b$ then the set $E = \{k \in \mathbb{N} : ka \leq b\} \neq \emptyset$ because $1 \in E$. Since for every $k \in E, ka \leq b \implies k \leq b/a, E$ is bounded. By completeness axiom, $E$ has a finite supremum, say $\sup E = s$. Since $E$ is a set of integers, $s \in E$. Set $n = s + 1$. Notice that $n \notin E \implies b < na$. \qed

Example 1.1. The supremum of $\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots\}$ is 1.

Example 1.2. The supremum of $\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \cdots\}$ is 1.

Proof. Let $M$ be an upper bound of $E$. We want to show that $M \geq 1$. If $M < 1$ then $1 - M > 0$ and $\frac{1}{1-M} > 0$. Then there exists $n \in \mathbb{N}$ such that $\frac{1}{1-M} < n \implies M < 1 - \frac{1}{n} = \frac{n-1}{n} \in E$. This contradicts the assumption that $M$ is an upper-bound. \qed

Theorem 1.2 (Density of Rationals). Let $a, b \in \mathbb{R}$ and $a < b$. Then there exists a $q \in \mathbb{Q}$ such that $a < q < b$.

Proof. Suppose that $a > 0$. By Archimedean Principle, choose $n \in \mathbb{N}$ satisfying

$$\max \left\{ \frac{1}{a}, \frac{1}{b-a} \right\} < n.$$ 

Observe that $1/n < a$ and $1/n < (b-a)$.

Consider the set $E = \{k \in \mathbb{N} : k/n \leq a\}$. $1 \in E \neq \emptyset$ and $E$ is bounded above by $na$. $\implies \sup E = s$ exists and $s \in E$. Take $q = \frac{s+1}{n}$. Since $s + 1 \notin E, q = \frac{s+1}{n} > a$.

On the other hand,

$$b = a + (b-a) > \frac{s}{n} + \frac{1}{n} = q.$$ 

If $a \leq 0$, by Archimedean Principle, there exists $k \in \mathbb{N}$ such that $-a < k$. Then $0 < a + k < b + k$. By the previous case we have $r \in \mathbb{Q}$ such that $a + k < r < b + k$. Therefore $r - k \in \mathbb{Q}$ and $a < r - k < b$. \qed

How are supremum and infimum of a set are related? For $E \subseteq \mathbb{R}$, we define $-E$ to be the set $-E = \{-x : x \in E\}$.

Theorem 1.3 (Reflection Theorem). Let $E \subseteq \mathbb{R}$ be a nonempty set.

(i) $E$ has a supremum $\iff -E$ has an infimum. Moreover, $- \sup E = \inf(-E)$.

(ii) $E$ has an infimum $\iff -E$ has a supremum. Moreover, $- \inf E = \sup(-E)$. 


Proof. We prove only (i). Suppose \( \sup E = s \). Then \( a \leq s \) for all \( a \in E \) \( \implies \) \( -s \leq -a \) for all \( a \in E \) \( \implies \) \( -s \) is a lower bound of \( -E \). We want to show that \( -s = \inf(-E) \). Let \( m \) be any lower bound of \( -E \). Then \( m \leq -a \) for all \( a \in E \) \( \implies \) \( a \leq -m \) for all \( a \in E \) \( \implies \) \( -m \) is an upper bound of \( E \) \( \implies \) \( s \leq -m \) \( \implies \) \( -s \geq m \). Thus 
\[
-\sup E = -s = \inf(-E).
\]

**Theorem 1.4** (Monotone Property). Suppose that \( A \subseteq B \) are nonempty subset of \( \mathbb{R} \).

(i) If \( B \) has a supremum, then \( \sup A \leq \sup B \).

(ii) If \( A \) has an infimum, then \( \inf A \geq \inf B \).

**Proof.** Since \( \sup B \) is an upper bound of \( B \), it is an upper bound of \( A \). Hence, by definition of supremum, \( \sup A \leq \sup B \). \( \square \)

### 1.4 Mathematical Induction

**Theorem 1.5** (Well-Ordering Principle). If \( E \) is a nonempty subset of \( \mathbb{N} \), then \( E \) has a least element, that is, \( \inf E \in E \).

**Proof.** The set \( -E \) is bounded above by \( -1 \). Thus \( \sup(-E) \) exists (by Completeness axiom) and \( \sup(-E) \in -E \). This implies \( \inf E = -\sup(-E) \in -(E) = E \). \( \square \)

**Theorem 1.6** (Principle of Mathematical Induction). Suppose for each \( n \in \mathbb{N} \) that \( A(n) \) is a proposition which satisfies the following two properties:

1. \( A(1) \) is true.

2. For every \( n \in \mathbb{N} \), if \( A(n) \) is true then \( A(n+1) \) is true.

Then \( A(n) \) is true for all \( n \in \mathbb{N} \).

**Proof.** Suppose that there are some \( n \in \mathbb{N} \) for which \( A(n) \) is not true. Then the set \( E = \{ n \in \mathbb{N} : A(n) \text{ is false} \} \neq \emptyset \). Then by the Well-Ordering Principle, \( E \) has a least element, say \( \inf E = x \) and \( x \in E \). Since \( 1 \notin E \), \( x > 1 \). So \( x - 1 > 0 \) and \( 1 \leq x - 1 \in \mathbb{N} \).

Since \( x - 1 < x \) and \( x \) is the least element of \( E \), the statement \( A(x-1) \) must be true. Then hypothesis (2) implies \( A(x) \) is true and \( x \notin E \), a contradiction. \( \square \)

**Example 1.3.** Prove that
\[
\sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.
\]
CHAPTER 1. THE REAL NUMBER SYSTEM

We can use Pascal’s triangle to expand \((a + b)^n\).

Some notations: \(0! = 1\), \(n! = 1 \cdot 2 \ldots (n - 1) \cdot n, n \in \mathbb{N}\). The binomial coefficient \(n\) over \(k\):

\[
\binom{n}{k} := \frac{n!}{(n-k)!k!}
\]

Lemma 1.7. If \(n, k \in \mathbb{N}\) and \(1 \leq k \leq n\), then

\[
\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}.
\]

Proof.

\[
\binom{n}{k-1} + \binom{n}{k} = \frac{n!k}{(n-k+1)!k!} + \frac{n!(n-k+1)}{(n-k+1)!k!} = \frac{n!(n+1)}{(n-k+1)!k!} = \binom{n+1}{k}.
\]

\[\square\]

Theorem 1.8 (Binomial Formula). If \(a, b \in \mathbb{R}, n \in \mathbb{N}\), then

\[(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k}b^k.\]

Proof.

\[
(a + b)^{n+1} = (a + b)(a + b)^n
\]

\[
= (a + b) \left( \sum_{k=0}^{n} \binom{n}{k} a^{n-k}b^k \right)
\]

\[
= \sum_{k=0}^{n} \binom{n}{k} a^{n-k+1}b^k + \sum_{k=0}^{n} \binom{n}{k} a^{n-k}b^{k+1}
\]

\[
= a^{n+1} + \sum_{k=1}^{n} \binom{n}{k} a^{n-k+1}b^k + \sum_{k=0}^{n-1} \binom{n}{k} a^{n-k}b^{k+1} + b^{n+1}
\]

\[
= a^{n+1} + \sum_{k=1}^{n} \binom{n}{k} a^{n-k+1}b^k + \sum_{k=1}^{n} \binom{n}{k-1} a^{n-k+1}b^k + b^{n+1}
\]

\[
= a^{n+1} + \sum_{k=1}^{n} \left( \binom{n}{k} + \binom{n}{k-1} \right) a^{n-k+1}b^k + b^{n+1}
\]

\[
= \sum_{k=0}^{n+1} \binom{n+1}{k} a^{n+1-k}b^k
\]

\[\square\]
Remark 1.1. If \( x > 1 \) and \( x \notin \mathbb{N} \), then there is an \( n \in \mathbb{N} \) such that \( n < x < n + 1 \).

\textbf{Proof.} By Archimedean Principle, the set \( E = \{ m \in \mathbb{N} : x < m \} \neq \emptyset \). By Well-Ordering Principle, \( E \) has a least element, say \( m_0 \). Then \( m_0 - 1 < x < m_0 \). Take \( n = m_0 - 1 \) to get \( n < x < n + 1 \). Observe that \( n \geq 1 \) and hence \( n \in \mathbb{N} \). \( \square \)

Remark 1.2. If \( n \in \mathbb{N} \) is not a perfect square, that is, \( n \neq m^2 \) for any \( m \in \mathbb{N} \), then \( \sqrt{n} \) is irrational.

\textbf{Proof.} Suppose that there is an \( n \in \mathbb{N} \) which is not a perfect square but \( \sqrt{n} \in \mathbb{Q} \). Write \( \sqrt{n} = p/q \) for some \( p, q \in \mathbb{N} \). By the previous remark there exists \( m_0 \in \mathbb{N} \) such that \( m_0 < \sqrt{n} < m_0 + 1 \).

The set \( E := \{ k \in \mathbb{N} : k\sqrt{n} \in \mathbb{Z} \} \neq \emptyset \) because \( q\sqrt{n} = p \in E \). By the Well-Ordering Principle, \( E \) has a least element, say \( n_0 \).

Now since \( 0 < \sqrt{n} - m_0 < 1 \), \( n_0(\sqrt{n} - m_0) < n_0 \). This implies \( n_0(\sqrt{n} - m_0) \notin E \). But,

\[ n_0(\sqrt{n} - m_0)\sqrt{n} = n_0n - n_0m_0\sqrt{n} \in \mathbb{Z} \]

implies \( n_0(\sqrt{n} - m_0) \in E \), a contradiction. \( \square \)

\section*{Exercises}

1. Let \( x, y, a \in \mathbb{R} \). Prove that \( x > y - \varepsilon \) for all \( \varepsilon > 0 \) if and only if \( x \geq y \).

2. Let \( E \subset \mathbb{R} \) be a nonempty set and \( \inf E \) exists. Prove that for any \( \varepsilon > 0 \) there exists a point \( a \in E \) such that \( \inf E \leq a < \inf E + \varepsilon \).

3. Let \( E \subset \mathbb{R} \) be a nonempty set. Assume that \( \inf E \) exists. Prove that \( -\inf E = \sup(\mathbb{E}) \).

4. 1.4.2 (a), 1.4.4 (b)

\section{1.5 Inverse Functions and Images}

\textbf{Definition 1.1}. A function \( f : X \to Y \) has an inverse function if and only if range \( f = Y \) and each \( y \in Y \) has a unique preimage \( x \in X \), that is, \( f(x) = y \).

\textbf{Definition 1.2}. Let \( X \) and \( Y \) be sets and \( f : X \to Y \).

i. \( f \) is said to be one-to-one (1 - 1 or an injection) if and only if \( x, x' \in X \) and \( f(x) = f(x') \Rightarrow x = x' \)

ii. \( f \) is said to be onto (or surjection) if and only if for each \( y \in Y \) there is an \( x \in X \) such that \( f(x) = y \).
iii. \( f \) is called bijection if and only if it is one-to-one and onto.

If \( f : X \to Y \) has an inverse, we will write \( f^{-1} : Y \to X \). In this case \( f(x) = y \iff x = f^{-1}(y) \).

**Theorem 1.9.** Let \( f : X \to Y \). Then the following statements are equivalent.

1. \( f \) has an inverse.
2. \( f \) one-to-one from \( X \) onto \( Y \).
3. There is a function \( g : Y \to X \) such that \( g(f(x)) = x \) for all \( x \in X \) and \( f(g(y)) = y \) for all \( y \in Y \).

**Proof.** (1) \( \implies \) (2). Assume that \( f \) has an inverse. Then range \( f = Y \). This implies \( f \) is onto. Also every \( y \in Y \) has a unique preimage in \( X \). This implies \( f \) is one-to-one.

(2) \( \implies \) (3). \( f \) is onto implies range \( f = Y \). This implies every \( y \in Y \) has a preimage in \( X \). Since \( f \) is one-to-one, the preimage is unique. Thus \( f \) has an inverse \( f^{-1} : Y \to X \). Take \( g = f^{-1} \).

(3) \( \implies \) (1). Let \( y \in Y \). Take \( x = g(y) \). Then \( x \in X \) and \( f(x) = f(g(y)) = y \). This implies that every \( y \in Y \) has an preimage in \( X \), that is, range \( f = Y \).

If \( x \) and \( x' \) are two preimages of \( y \in Y \), then \( f(x) = y = f(x') \). Then it follows that \( x = g(f(x)) = g(f(x')) = x' \). Hence the preimage is unique. Thus \( f \) has an inverse. \( \Box \)

**Theorem 1.10.** Let \( f : X \to Y \) has an inverse function. Then the inverse function is unique.

**Proof.** Suppose that \( g : Y \to X \) and \( h : Y \to X \) are two inverse functions of \( f \). We want to show that \( g(y) = h(y) \) for every \( y \in Y \). Take a \( y \in Y \). Then there is an \( x \in X \) such that \( f(x) = y \). Now, \( g(y) = g(f(x)) = x = h(f(x)) = h(y) \). Thus \( g = h \). \( \Box \)

**Remark 1.3.** Let \( I \subset \mathbb{R} \) be an interval and \( f : I \to R \). If \( f'(x) > 0 \), then \( f \) is injective on \( I \).

**Proof.** Recall that \( f \) is called strictly increasing on \( I \) if for \( x,x' \in I, x < x' \implies f(x) < f(x') \). Since \( f'(x) > 0 \) on \( I \), \( f \) is strictly increasing on \( I \). Let \( x,x' \in I \). If \( x \neq x' \) then either \( x < x' \) or \( x > x' \). Since \( f \) is strictly increasing, either \( f(x) < f(x') \) or \( f(x) > f(x') \). Hence \( x \neq x' \) implies \( f(x) \neq f(x') \). Thus \( f \) is one-to-one. \( \Box \)

If \( f : X \to Y \) has an inverse function \( f^{-1} : Y \to X \) then \( y = f(x) \) can be solved for \( x \).
Definition 1.3. Let $f : X \to Y$. The image of a set $E \subseteq X$ under $f$ is the set

$$f(E) := \{ y \in Y : y = f(x) \text{ for some } x \in E \}.$$

The inverse image of a set $E \subseteq Y$ under $f$ is the set

$$f^{-1}(E) := \{ x \in X : f(x) = y \text{ for some } y \in Y \}.$$

Example 1.4. Let $E = (0, 1)$ and $f(x) = x^2$. Find $f(E)$ and $f^{-1}(E)$.

Example 1.5. Let $E = (0, \infty)$ and $f : E \to \mathbb{R}$ be defined by $f(x) = e^{1/x}$. Find $f(E)$ and $f^{-1}$ on $f(E)$.

Definition 1.4. Let $\mathcal{E} = \{ E_\alpha \}_{\alpha \in A}$ be a collection of sets. Then

$$\bigcup_{\alpha \in A} E_\alpha := \{ x : x \in E_\alpha \text{ for some } \alpha \in A \}$$

and

$$\bigcap_{\alpha \in A} E_\alpha := \{ x : x \in E_\alpha \text{ for all } \alpha \in A \}$$

We have, $\bigcup_{x \in \{0,1\}} = [0,1)$ and $\bigcap_{x \in \{0,1\}} [0, x) = 0$.

Theorem 1.11 (DeMorgan’s Laws). Let $X$ be a set and $\{ E_\alpha \}_{\alpha \in A}$ be a collection of subsets of $X$. Then

$$\left( \bigcup_{\alpha \in A} E_\alpha \right)^c = \bigcap_{\alpha \in A} E_\alpha^c$$

and

$$\left( \bigcap_{\alpha \in A} E_\alpha \right)^c = \bigcup_{\alpha \in A} E_\alpha^c$$

Note that $E^c = X \setminus E$.

Proof. Let $x \in \left( \bigcup_{\alpha \in A} E_\alpha \right)^c$.

$$\implies x \in X \text{ and } x \notin \bigcup_{\alpha \in A} E_\alpha.$$

$$\implies x \notin E_\alpha \text{ for all } \alpha \in A.$$

$$\implies x \in E_\alpha^c \text{ for all } \alpha \in A.$$

$$\implies x \in \bigcap_{\alpha \in A} E_\alpha^c.$$
Theorem 1.12. Let $X$ and $Y$ be sets and $f : X \to Y$.

i. If $\{E_\alpha\}_\alpha \in A$ is a collection of subsets of $X$, then

$$f \left( \bigcup_{\alpha \in A} E_\alpha \right) = \bigcup_{\alpha \in A} f(E_\alpha) \quad \text{and} \quad f \left( \bigcap_{\alpha \in A} E_\alpha \right) = \bigcap_{\alpha \in A} f(E_\alpha).$$

ii. If $B$ and $C$ are subsets of $X$, then $f(C \setminus B) \supseteq f(C) \setminus f(B)$.

iii. If $\{E_\alpha\}_\alpha \in A$ is a collection of subsets of $Y$, then

$$f^{-1} \left( \bigcup_{\alpha \in A} E_\alpha \right) = \bigcup_{\alpha \in A} f^{-1}(E_\alpha) \quad \text{and} \quad f^{-1} \left( \bigcap_{\alpha \in A} E_\alpha \right) = \bigcap_{\alpha \in A} f^{-1}(E_\alpha).$$

iv. If $B$ and $C$ are subsets of $Y$, then $f^{-1}(C \setminus B) = f^{-1}(C) \setminus f^{-1}(B)$.

v. If $E \subseteq f(X)$, then $f(f^{-1}(E)) = E$, but if $E \subseteq X$, then $E \subseteq f^{-1}(f(E))$.

Proof. (i).

$$y \in f \left( \bigcup_{\alpha \in A} E_\alpha \right) \iff y = f(x) \text{ for some } x \in \bigcup_{\alpha \in A} E_\alpha.$$  

$$\iff y = f(x) \text{ for some } x \in E_\alpha \text{ and } \alpha \in A.$$  

$$\iff y \in f(E_\alpha) \text{ for some } \alpha \in A.$$  

$$\iff y \in \bigcup_{\alpha \in A} f(E_\alpha).$$

(ii). Let $y \in f(C) \setminus f(B)$.

$$\implies y \in f(C) \text{ and } y \notin f(B).$$  

$$\implies y = f(c) \text{ for some } c \in C \text{ but } y \neq f(b) \text{ for any } b \in B.$$  

$$\implies y = f(c) \text{ for some } c \in C \setminus B.$$  

$$\implies y \in f(C \setminus B).$$

(v). Let $y \in f(f^{-1}(E))$. Then $y = f(x)$ for some $x \in f^{-1}(E)$. By definition, $x = f^{-1}(y')$ for some $y' \in E$. But $y = f(x) = f(f^{-1}(y')) = y'$ implies $y' \in E$. Thus $f(f^{-1}(E)) \subseteq E$.

For the other way around, let $y \in E$. Then $y = f(f^{-1}(y)) \in f(f^{-1}(E))$. Thus $E \subseteq f(f^{-1}(E))$. Therefore we have $f(f^{-1}(E)) = E$ where $E \subseteq f(X)$.

For the second part, let $y \in E$. Then $y = f^{-1}(f(y)) \in f^{-1}(f(E))$. Hence, $E \subseteq f^{-1}(f(E))$. □
1.6 Countable and Uncountable Sets

Definition 1.5.
A set \( E \) is said to be finite if and only if either \( E = \emptyset \) or there exists a bijective function from \( \{1, 2, \ldots, n\} \) to \( E \) for some \( n \in \mathbb{N} \).

A set \( E \) is said to be countable if and only if there exists a bijective function from \( \mathbb{N} \) to \( E \). \( E \) is said to be at most countable if it is finite or countable.

A set \( E \) is said to be uncountable if and only if \( E \) neither finite nor countable.

Remark 1.4 (Cantor’s Diagonalization Argument). The interval \((0, 1)\) is uncountable.

Proof. Suppose that \((0, 1)\) is countable. Then there is a bijection \( f : \mathbb{N} \to (0, 1) \), that is, \( f(\mathbb{N}) = (0, 1) \). Then by using decimal expansion, we can write

\[
\begin{align*}
  f(1) &= 0.a_{11}a_{12}a_{13} \cdots \\
  f(2) &= 0.a_{21}a_{22}a_{23} \cdots \\
  f(3) &= 0.a_{31}a_{32}a_{33} \cdots \\
  f(4) &= 0.a_{41}a_{42}a_{43} \cdots \\
  \vdots &= \ldots
\end{align*}
\]

where none of these expansions terminate in 9’s. For example 0.123999\ldots = 0.124.

Now the number \( x = 0.b_1b_2b_3 \cdots \) where \( b_k := \begin{cases} a_{kk} + 1 & \text{if } a_{kk} \leq 5 \\ a_{kk} - 1 & \text{if } a_{kk} > 5. \end{cases} \) Clearly \( x \in (0, 1) \). Since \( f(\mathbb{N}) = (0, 1) \) there is an \( n \in \mathbb{N} \) such that \( f(n) = x \), that is,

\[
0.b_1b_2 \cdots b_n \cdots = 0.a_{n1}a_{n2} \cdots a_{nn} \cdots
\]

Since the decimals on either side don’t terminate in 9’s, we have to have

\[
a_{nn} = b_n = a_{nn} \pm 1
\]

which is a contradiction.

Lemma 1.13. A non-empty set is at most countable if and only if there is a surjective function \( f : \mathbb{N} \to E \).

Proof. Reading exercise.

Theorem 1.14. Suppose that \( A \) and \( B \) are sets and \( A \subseteq B \).

i. If \( B \) is at most countable, then \( A \) is at most countable.

ii. If \( A \) is uncountable, then \( B \) is uncountable.
Proof. (i). Let $B$ be at most countable. Then there is a surjective function $f : \mathbb{N} \to B$. If $A = B$ then $f$ is automatically a surjective function from $\mathbb{N}$ to $A$. If $A \subseteq B$, then define $g : \mathbb{N} \to A$ by

$$ g(n) = \begin{cases} f(n) & \text{if } f(n) \in A \\ a_0 \text{ for some } a_0 \in A & \text{if } f(n) \notin A. \end{cases} $$

Then $g : \mathbb{N} \to A$ is surjective. So $A$ is at most countable.

(ii). If $B$ is at most countable then by (i), $A$ is at most countable, which is a contradiction. So $B$ is uncountable if $A$ is.

Since $(0, 1)$ is uncountable and $(0, 1) \subset \mathbb{R}$, $\mathbb{R}$ is uncountable.

**Theorem 1.15.**

1. $\mathbb{N} \times \mathbb{N}$ is countable. In particular, if $A$ and $B$ are at most countable sets, then $A \times B$ is at most countable.

2. If $\{A_i\}_{i=1}^\infty$ is a collection of at most countable sets, then $E = \bigcup_{i=1}^\infty A_i$ is at most countable.

Proof. 1. Let us arrange the points of $\mathbb{N} \times \mathbb{N}$ as following,

$(1, 1) \ (1, 2) \ (1, 3) \ (1, 4) \ (1, 5) \ \ldots$
$(2, 1) \ (2, 2) \ (2, 3) \ (2, 4) \ \ldots$
$(3, 1) \ (3, 2) \ (3, 3) \ \ldots$
$(4, 1) \ (4, 2) \ \ldots$
$(5, 1) \ \ldots$
\[ \ldots \]

Let us arrange the natural numbers similarly,

$1 \ 3 \ 6 \ 10 \ 15 \ 21 \ \ldots$
$2 \ 5 \ 9 \ 14 \ 20 \ \ldots$
$4 \ 8 \ 13 \ 19 \ \ldots$
$7 \ 12 \ 18 \ \ldots$
$11 \ 17 \ \ldots$
$16 \ \ldots$
\[ \ldots \]

Now define a function $g : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ that sends a number $n \in \mathbb{N}$ to the corresponding point in $\mathbb{N} \times \mathbb{N}$ in these two arrangements.

To show that $A \times B$ is at most countable define $f : \mathbb{N} \times \mathbb{N} \to A \times B$ by

$$ f(m, n) = (\phi(m), \psi(n)) $$
where φ : N → A and ψ : N → B are surjective functions. Hence f ◦ g : N → A × B is surjective and therefore A × B is at most countable.

2. To show that $E = \bigcup_{i \in \mathbb{N}} A_i$ is at most countable, define $f : \mathbb{N} \times \mathbb{N} \to E$ by $f(i, j) = \phi_i(j)$ where $\phi_i : \mathbb{N} \to A_i$ is a surjective function for all $i \in \mathbb{N}$. Then $f$ is surjective (why?). And, now $f \circ g : \mathbb{N} \to E$ is surjective. Hence $E$ is at most countable.

**Remark 1.5.** The sets $\mathbb{Z}$ and $\mathbb{Q}$ are countable. The set of irrationals is uncountable.

**Proof.** Since $\mathbb{Z} = \mathbb{N} \cup -\mathbb{N} \cup \{0\}$, $\mathbb{Z}$ is countable. Similarly $\mathbb{Q} = \bigcup_{n=1}^{\infty} \{\frac{p}{n} : p \in \mathbb{Z}\}$ implies $\mathbb{Q}$ is countable. If the set of irrational were countable then $\mathbb{R} = \mathbb{Q} \cup \{\text{irrationals}\}$ would be countable. So the irrationals are uncountable.

**Exercises**

1. Let $I \subset \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$. Prove that if $f'(x) < 0$, then $f$ is injective on $I$.

2. Let $f(x) = x^2 + 1$. Find $f(E)$ and $f^{-1}(E)$, where $E = [1, 2]$.

3. 1.5.3 (c, d)

4. If $\{E_\alpha\}_{\alpha \in A}$ is a collection of subsets of $X$, then prove that

$$f \left( \bigcap_{\alpha \in A} E_\alpha \right) = \bigcap_{\alpha \in A} f(E_\alpha).$$

5. Let $f : X \to Y$ and $E \subseteq X$. Construct an example where $E \neq f^{-1}(f(E))$.

6. Let $f : A \to B$ and $g : B \to C$. Then the composition $g \circ f : A \to C$ is defined by $g \circ f(x) = g(f(x))$. Prove that if $f$ and $g$ are one-to-one, then $g \circ f$ is one-to-one.

7. 1.6.7 is a good Project.
Chapter 2

Sequences of Real Numbers

A sequence is a function defined on \( \mathbb{N} \). If \( f : \mathbb{N} \to \mathbb{R} \) is a sequence, then we write \( f(n) = x_n \) and call \( x_n \) the \( n \)-th term of the sequence. Instead of using a function, we will rather use the terms to denote a sequence. So we will use one of the following notations to denote a sequence:

\[
\{x_1, x_2, x_3, \cdots \} \text{ or } \{x_n\}_{n \in \mathbb{N}} \text{ or } \{x_n\}_{n=1}^{\infty} \text{ or } \{x_n\}.
\]

Example 2.1. The function \( f : \mathbb{N} \to \mathbb{R} \) defined by \( f(n) = 1/n \) is the sequence \( \{1, 1/2, 1/3, \cdots \} \).

2.1 Limit of a Sequence

Definition 2.1. A sequence of real numbers \( \{x_n\} \) is said to converge to a real number \( l \in \mathbb{R} \) if for every \( \varepsilon > 0 \) there exists an \( n_0 \in \mathbb{N} \) such that for every \( n \geq n_0 \) we have \( |x_n - l| < \varepsilon \).

We will call \( l \) the limit of the sequence \( \{x_n\} \).

Note that in the above definition the choice of \( n_0 \) usually depends on \( \varepsilon \), that means \( n_0 \) changes with the different choice of \( \varepsilon \). When a sequence \( \{x_n\} \) converges to a number \( l \) we will write it in one of the following ways:

1. \( \{x_n\} \) converges to \( l \).
2. \( x_n \) converges to \( l \).
3. \( x_n \to l \) as \( n \to \infty \).
4. \( \lim_{n \to \infty} x_n = l \)

The following statements follow immediately from the definition:
1. A sequence \( \{x_n\} \) converges to \( l \) if and only if \( \{x_n - l\} \) converges to 0.

2. \( x_n \to 0 \) as \( n \to \infty \) if and only if \( |x_n| \to 0 \) as \( n \to \infty \).

**Example 2.2.** The sequence \( \{\frac{1}{n}\} \) converges to 0.

**Proof.** Let \( \varepsilon > 0 \) be given. By Archimedean Principle, choose \( n_0 \in \mathbb{N} \) such that \( n_0 > \frac{1}{\varepsilon} \). Then for every \( n \geq n_0 \),

\[
\left| \frac{1}{n} - 0 \right| = \frac{1}{n} \leq \frac{1}{n_0} < \varepsilon.
\]

\[\square\]

**Example 2.3.** The sequence \( \{(−1)^n\} \) has no limit.

**Proof.** Suppose that \( (−1)^n \to l \). Then for given \( \varepsilon = 1 \), there exists \( n_0 \in \mathbb{N} \) such that \( |(−1)^n - l| < 1 \) whenever \( n \geq n_0 \).

For \( n \) odd, \( |−1 − l| < 1 \), and for \( n \) even \( |1 − l| < 1 \). Hence,

\[
2 = |1 + 1| = |(−1 − l) + (1 + l)| \leq |1 − l| + |1 + l| < 1 + 1 = 2
\]

which is a contradiction. \[\square\]

**Theorem 2.1.** If a sequence \( \{x_n\} \) has a limit, it is unique.

**Proof.** Suppose that \( x_n \to a \) and \( x_n \to b \) as \( n \to \infty \). Then we want to show that \( a = b \). Let \( \varepsilon > 0 \) be given. Then

\[
x_n \to a \implies \text{there is an } n_0 \in \mathbb{N} \text{ such that } |x_n - a| < \varepsilon/2 \text{ whenever } n \geq n_0.
\]

\[
x_n \to b \implies \text{there is an } n_1 \in \mathbb{N} \text{ such that } |x_n - b| < \varepsilon/2 \text{ whenever } n \geq n_1.
\]

For \( n \geq \max\{n_0, n_1\} \),

\[
|a - b| \leq |x_n - a| + |x_n - b| < \varepsilon/2 + \varepsilon/2 = \varepsilon.
\]

That is, \( |a - b| < \varepsilon \) for all \( \varepsilon > 0 \). Hence \( a - b = 0 \) and \( a = b \). \[\square\]

**Definition 2.2.** A subsequence of a sequence \( \{x_n\}_{n \in \mathbb{N}} \) is a sequence of the form \( \{x_{n_k}\}_{k \in \mathbb{N}} \), where \( n_k \in \mathbb{N} \) and \( n_1 < n_2 < \cdots \). That is, a subsequence of a sequence \( \{x_n\} \) is a sequence which is a subset of \( \{x_n\} \).

**Theorem 2.2.** If a sequence \( \{x_n\} \) converges, then every subsequence of \( \{x_n\} \) converges to the same limit.
Proof. Let \( x_n \to a \) as \( n \to \infty \) and \( \{x_{n_k}\} \) be any subsequence of \( \{x_n\} \). We want to show that \( x_{n_k} \to a \) as \( k \to \infty \). Let \( \varepsilon > 0 \) and \( n_0 \in \mathbb{N} \) such that \( |x_n - a| < \varepsilon \) whenever \( n \geq n_0 \). Notice that \( k \geq n_0 \) implies \( n_k \geq n_0 \). Hence it follows that \( |x_{n_k} - a| < \varepsilon \) whenever \( k \geq n_0 \). Thus \( x_{n_k} \to a \) as \( k \to \infty \).

Definition 2.3. A sequence \( \{x_n\} \) is said to be bounded above if and only if the underlying set \( \{x_n : n \in \mathbb{N}\} \) is bounded above and it is said to be bounded below if the underlying set is bounded below. A sequence is called bounded if it is both bounded above and bounded below.

A sequence \( \{x_n\} \) is bounded if and only if there is an \( M \in \mathbb{R} \) such that \( x_n \leq M \) for all \( n \in \mathbb{N} \). Similarly, \( \{x_n\} \) is bounded below if and only if there is an \( m \in \mathbb{R} \) such that \( m \leq x_n \) for all \( n \in \mathbb{N} \). And \( \{x_n\} \) is bounded if and only if there is a number \( M > 0 \) such that \( |x_n| < M \) for all \( n \in \mathbb{N} \).

Theorem 2.3. If a sequence \( \{x_n\} \) is convergent, then it is bounded.

Proof. Let \( \varepsilon = 1 \) and \( x_n \to l \) as \( n \to \infty \). Then by definition, there exists an \( n_0 \in \mathbb{N} \) such that

\[ |x_n - l| < 1 \text{ whenever } n \geq n_0. \]

which implies

\[ |x_n| < 1 + |l| \text{ whenever } n \geq n_0. \]

Take \( M = \max\{x_1, x_2, \ldots, x_{n_0}, 1 + |l|\} \). Hence \( |x_n| < M \) for all \( n \in \mathbb{N} \).}

\[ \square \]

2.2 Limit Theorems

Sometimes if we know the convergence or divergence of a sequence we can use it to determine the convergence or divergence of another sequence.

Theorem 2.4 (Squeeze Theorem). Suppose that the sequences \( \{x_n\} \) and \( \{y_n\} \) converge to the same limit \( l \). If \( \{w_n\} \) is a sequence such that

\[ x_n \leq w_n \leq y_n \text{ for } n \geq n_0 \]

for some \( n_0 \in \mathbb{N} \) then \( \{w_n\} \) converges to \( l \).

Proof. Let \( \varepsilon > 0 \). We want to show that \( w_n \to l \) as \( n \to \infty \).

\[ x_n \to l \text{ as } n \to \infty \implies \text{ there exists } n_1 \in \mathbb{N} \text{ such that } |x_n - l| < \varepsilon \text{ whenever } n \geq n_1. \]

\[ \implies l - \varepsilon < x_n < l + \varepsilon \text{ whenever } n \geq n_1. \]

\[ y_n \to l \text{ as } n \to \infty \implies \text{ there exists } n_2 \in \mathbb{N} \text{ such that } |y_n - l| < \varepsilon \text{ whenever } n \geq n_2. \]

\[ \implies l - \varepsilon < y_n < l + \varepsilon \text{ whenever } n \geq n_2. \]
Thus we have
\[ l - \varepsilon < x_n < w_n < y_n < l + \varepsilon \] whenever \( n \geq \max\{n_0, n_1, n_2\} \).
\[ \implies -\varepsilon < w_n - l < \varepsilon \] whenever \( n \geq \max\{n_0, n_1, n_2\} \).
\[ \implies w_n \to l \text{ as } n \to \infty. \]

**Theorem 2.5.** If \( x_n \to 0 \) as \( n \to \infty \) and \( \{y_n\} \) is bounded, then \( x_n y_n \to 0 \) as \( n \to \infty \).

**Proof.** Since \( \{y_n\} \) is bounded, there exists an \( M \in \mathbb{R} \) such that \( |y_n| \leq M \) for all \( n \in \mathbb{N} \). We have
\[ 0 \leq |x_n y_n| = |x_n||y_n| \leq M|x_n| \]
for all \( n \in \mathbb{N} \). Since \( x_n \to 0 \) as \( n \to \infty \), \( |x_n| \to 0 \) as \( n \to \infty \) and hence \( M|x_n| \to 0 \) as \( n \to \infty \). Hence by the Squeeze Theorem \( |x_n y_n| \to 0 \) and hence \( x_n y_n \to 0 \) as \( n \to \infty \). \( \square \)

**Theorem 2.6.** Let \( E \subset \mathbb{R} \) has a finite supremum. Then there is a sequence \( \{x_n\} \) in \( E \) such that \( x_n \to \sup E \) as \( n \to \infty \). Similarly, If \( E \) has a finite infimum, then there is a sequence \( \{y_n\} \) in \( E \) such that \( y_n \to \inf E \) as \( n \to \infty \).

**Proof.** For every \( n \in \mathbb{N} \), by the Approximation Property of supremum, there exists \( x_n \in E \) such that \( \sup E - \frac{1}{n} < x_n \leq \sup E \).

By the Squeeze Theorem \( x_n \to \sup E \) as \( n \to \infty \). \( \square \)

**Theorem 2.7.** Suppose that \( \{x_n\} \) and \( \{y_n\} \) are convergent sequences and \( \alpha \in \mathbb{R} \). Then
\[ 1. \lim_{n \to \infty} (x_n + y_n) = \lim_{n \to \infty} x_n + \lim_{n \to \infty} y_n \\
2. \lim_{n \to \infty} (\alpha x_n) = \alpha \lim_{n \to \infty} x_n \\
3. \lim_{n \to \infty} (x_n y_n) = \lim_{n \to \infty} x_n \lim_{n \to \infty} y_n \\
4. \lim_{n \to \infty} \frac{x_n}{y_n} = \frac{\lim_{n \to \infty} x_n}{\lim_{n \to \infty} y_n} \text{ whenever } y_n \neq 0 \text{ and } \lim_{n \to \infty} y_n \neq 0. \]

**Proof.** Let \( \lim_{n \to \infty} x_n = a \) and \( \lim_{n \to \infty} y_n = b \). Let \( \varepsilon > 0 \).

1. We want to prove that \( \lim_{n \to \infty} (x_n + y_n) = a + b \).
\[ x_n \to a \implies \text{there exists } n_1 \in \mathbb{N} \text{ such that } |x_n - a| < \varepsilon \text{ whenever } n \geq n_1. \]
\[ y_n \to b \implies \text{there exists } n_2 \in \mathbb{N} \text{ such that } |y_n - a| < \varepsilon \text{ whenever } n \geq n_2. \]

Now,
\[ |(x_n + y_n) - (a + b)| \leq |x_n - a| + |y_n - b| < \varepsilon/2 + \varepsilon/2 = \varepsilon \]
whenever \( n \geq \max\{n_1, n_2\} \).
2. We want to prove that \( \lim_{n \to \infty} (\alpha x_n) = \alpha a. \) Let \( \varepsilon > 0. \) Then
\[
x_n \to a \implies \text{there exists } n_0 \in \mathbb{N} \text{ such that } |x_n - a| < \varepsilon/|\alpha| \text{ whenever } n \geq n_0.
\]
Now
\[
|\alpha x_n - \alpha a| = |\alpha||x_n - a| < |\alpha|\frac{\varepsilon}{|\alpha|} = \varepsilon
\]
whenever \( n \geq n_0. \)

3. Since the convergent sequence is bounded, there is an \( M \in \mathbb{R} \) such that \(|y_n| \leq M\) for all \( n \in \mathbb{N}. \) Let \( \varepsilon > 0. \) There exists \( n_1, n_2 \in \mathbb{N} \) such that \(|x_n - a| < \varepsilon/2M \) whenever \( n \geq n_1 \) and \(|y_n - b| < \varepsilon/2|a| \) whenever \( n \geq n_2. \)
\[
|x_n y_n - ab| = |(x_n y_n - ay_n) + (ay_n - ab)|
\]
\[
\leq |x_n - a||y_n| + |a||y_n - b|
\]
\[
< \frac{\varepsilon}{2M} M + |a|\frac{\varepsilon}{2|a|} = \varepsilon
\]

4. Exercise.

\[\square\]

**Example 2.4.** Evaluate \( \lim_{n \to \infty} \frac{2n^3 - 5n + 8}{3n^3 + 7}. \)

**Definition 2.4.** Let \( \{x_n\} \) be sequence. We say that \( \{x_n\} \) diverges to \( \infty, \) written \( \lim_{n \to \infty} x_n = \infty \) if and only if for each \( M \in \mathbb{R} \) there is \( n_0 \in \mathbb{N} \) such that \( x_n > M \) whenever \( n \geq n_0. \) And, we say that \( \{x_n\} \) diverges to \( -\infty, \) written \( \lim_{n \to \infty} x_n = -\infty \) if and only if for each \( M \in \mathbb{R} \) there is \( n_0 \in \mathbb{N} \) such that \( x_n < M \) whenever \( n \geq n_0. \)

**Theorem 2.8 (Comparison Theorem).** Suppose that \( x_n \) and \( y_n \) are convergent se-
quence. If there is an \( n_0 \in \mathbb{N} \) such that \( x_n \leq y_n \) for \( n \geq n_0, \) then \( \lim_{n \to \infty} x_n \leq \lim_{n \to \infty} y_n. \)

**Proof.** Let \( x_n \to a \) and \( y_n \to b \) as \( n \to \infty. \) We want to show that \( a \leq b. \) If \( a > b, \) then take \( \varepsilon = a - b. \) Take \( \varepsilon = \frac{a-b}{2}. \) Then there is an \( n_1 \in \mathbb{N} \) such that \(|x_n - a| < \varepsilon \) and \(|y_n - b| < \varepsilon \) whenever \( n \geq n_1. \) Then for \( n \geq n_1 \)
\[
x_n > a - \varepsilon = a - \frac{a-b}{2} = b + \frac{a-b}{2} = b + \varepsilon > y_n.
\]
This is a contradiction to the hypothesis. \[\square\]

**Remark 2.1.** If \( \{x_n\} \) and \( \{y_n\} \) are convergent then
\[
x_n < y_n, \text{ for } n \geq n_0 \implies \lim_{n \to \infty} x_n \leq \lim_{n \to \infty} y_n
\]
but not
\[
\lim_{n \to \infty} x_n < \lim_{n \to \infty} y_n.
\]
For example, think of \( \{\frac{1}{n^2}\} \) and \( \{\frac{1}{n}\}. \)
Exercises

1. Give an example to show that every bounded sequence may not converge.

2. 2.1.1 (a).

3. Consider the set $E = (1, 2) \subset \mathbb{R}$. Then $\inf E = 1$. Construct a sequence $x_n \in (1, 2)$ such that $x_n \to 1$ as $n \to \infty$.

4. Suppose that $\{x_n\}$ is a sequence of non-negative real numbers and $x_n \to x$ as $n \to \infty$. Prove that $\sqrt{x_n} \to \sqrt{x}$ as $n \to \infty$.

5. Let $r \in \mathbb{R}$. Prove that there exists a sequence of rational numbers $\{r_n\}$ such that $r_n \to r$ as $n \to \infty$.

6. Let $x_n \to l$ and $y_n \to l$ as $n \to \infty$. If $x_n \leq w_n \leq y_n$ for all $n \in \mathbb{N}$, prove that $w_n \to l$ as $n \to \infty$.

2.3 Bolzano-Weierstrass Theorem

Definition 2.5. We say that a sequence \( \{x_n\} \) is

i. increasing if $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$.

ii. strictly increasing if $x_n < x_{n+1}$ for all $n \in \mathbb{N}$.

iii. decreasing if $x_n \geq x_{n+1}$ for all $n \in \mathbb{N}$.

iv. strictly decreasing if $x_n > x_{n+1}$ for all $n \in \mathbb{N}$.

v. monotone if and only if it is either increasing or decreasing.

Theorem 2.9 (Monotone Convergence Theorem).

1. If $\{x_n\}$ is increasing and bounded above, then $\{x_n\}$ converges to a finite limit.

2. If $\{x_n\}$ is decreasing and bounded below, then $\{x_n\}$ converges to a finite limit.

Proof.

1. Suppose that $\{x_n\}$ is increasing and bounded above. Then by the Completeness Axiom, supremum of $\{x_n : n \in \mathbb{N}\}$ exists, say $a = \sup\{x_n : n \in \mathbb{N}\}$. Let $\varepsilon > 0$. Then by the Approximation Property of supremum, there exists $n_0 \in \mathbb{N}$ such that

   $$a - \varepsilon < x_{n_0} \leq a.$$

   Since $\{x_n\}$ is increasing

   $$a - \varepsilon < x_n \leq a$$

   whenever $n \geq n_0$. 
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2. Exercise.

Example 2.5. If $|a| < 1$, then $a^n \to 0$ as $n \to \infty$.  

Proof. Since $|a| < 1$, $|a|^{n+1} < |a|^n$ for all $n \in \mathbb{N}$. This implies that the sequence $\{|a|^n\}$ is decreasing. Observe that it is bounded below by 0. Hence the Monotone Convergence Theorem implies that $\{|a|^n\}$ converges. Let $\lim_{n \to \infty} |a|^n = l$. We want to show that $l = 0$. Write

$$|a|^{n+1} = |a| \cdot |a|^n \Rightarrow \lim_{n \to \infty} |a|^{n+1} = \lim_{n \to \infty} |a| \cdot |a|^n \Rightarrow l = |a| \cdot l \Rightarrow l = 0.$$  

Example 2.6. If $a > 0$, then $a^{1/n} \to 1$ as $n \to \infty$.

Proof. Reading Exercise.  

Definition 2.6. A sequence of sets $\{I_n\}$ is said to be nested if and only if $I_n \supseteq I_{n+1}$ for all $n \in \mathbb{N}$, that is,

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots.$$  

Theorem 2.10 (Nested Interval Property). If $\{I_n\}$ is a nested sequence of nonempty closed bounded intervals, then $E = \cap_{n=1}^{\infty} I_n$ is nonempty. Moreover, if the length of these intervals satisfy $|I_n| \to 0$ as $n \to \infty$, then $E$ is a single point.

Proof. Let $I_n = [a_n, b_n]$. Since $\{I_n\}$ is nested, we have

$$a_1 \leq a_2 \leq a_3 \leq \cdots \leq b_3 \leq b_2 \leq b_1.$$  

Thus the sequence of left end points $\{a_n\}$ is a bounded increasing sequence and the sequence of right end points $\{b_n\}$ is a bounded decreasing sequence. By the Monotone Convergence Theorem $\{a_n\}$ and $\{b_n\}$ are convergent. Let $a_n \to a$ and $b_n \to b$. Since $a_n \leq b_n$ for all $n \in \mathbb{N}$, $a \leq b$ and hence $a_n \leq a \leq b \leq b_n$ for all $n \in \mathbb{N}$. This implies $[a, b] \subseteq [a_n, b_n]$ for all $n \in \mathbb{N}$. Hence $\cap_{n=1}^{\infty} I_n \supseteq [a, b] \neq \emptyset$.  

If $|I_n| \to 0$ then $b_n - a_n \to 0$, that is, $\lim_{n \to \infty} (b_n - a_n) = 0 \Rightarrow b - a = \lim_{n \to \infty} b_n - \lim_{n \to \infty} a_n = 0$. Thus $a = b$ and $[a, b] = \{a\}$. 

Remark 2.2.

1. If the intervals in the nested sequence $\{I_n\}$ are not closed, then the conclusion of the above might not hold.
2. If the intervals in the nested sequence \( \{I_n\} \) are not bounded, then the conclusion of the above might not hold.

**Proof.** Here are the examples.

1. Take \( I_n = (0, 1/n) \). Then \( \bigcap_{n=1}^{\infty} I_n = \bigcap_{n=1}^{\infty} (0, 1/n) = \emptyset \) (why?).

2. Take \( I_n = [1, \infty) \). Then \( \bigcap_{n=1}^{\infty} I_n = \bigcap_{n=1}^{\infty} [n, \infty) = \emptyset \) (why?).

**Theorem 2.11** (Bolzano-Weierstrass Theorem). Every bounded sequence of real numbers has a convergent subsequence.

**Proof.** Let \( \{x_n\} \) bounded sequence. Let \( a \) be a lower bound and \( b \) be an upper bound of \( \{x_n\} \). Then \( x_n \in [a, b] \) for all \( n \in \mathbb{N} \). Split \( [a, b] \) into two halves \( [a, a+b/2] \) and \( [a+b/2, b] \). At least one of these intervals contains infinitely many terms of \( \{x_n\} \). Call it \( I_1 \) and choose \( n_1 \) such that \( x_{n_1} \in I_1 \). Observe that length of \( I_1 \) is \( b-a \).

Continue this process to get \( I_k \) for every \( k \in \mathbb{N} \) such that \( I_k \) contains infinitely many terms of \( x_n \). Then choose \( n_k \) such that \( n_k > n_{k-1} \) and \( x_{n_k} \in I_k \). Observe that the length of \( I_k \) is \( b-a \).

Thus we have a nested sequence of nonempty closed and bounded intervals

\[ I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots \]

where \( |I_k| = \frac{b-a}{2^k} \to 0 \) as \( k \to \infty \). This implies \( \bigcap_{k=1}^{\infty} I_k \) contains a single point. Let \( x \in \bigcap_{k=1}^{\infty} I_k \). Now,

\[ 0 \leq |x_{n_k} - x| \leq |I_k| = \frac{b-a}{2^k}. \]

Since \( \frac{b-a}{2^k} \to 0 \) as \( k \to \infty \), by the Squeeze Theorem \( |x_{n_k} - x| \to 0 \). Thus we have a convergent subsequence \( \{x_{n_k}\} \). \( \square \)

### 2.4 Cauchy Sequences

**Definition 2.7.** A sequence of real numbers \( \{x_n\} \) is called a Cauchy sequence if and only if for every \( \varepsilon > 0 \) there is an \( n_0 \in \mathbb{N} \) such that

\[ |x_n - x_m| < \varepsilon \text{ whenever } n, m \geq n_0. \]

**Theorem 2.12.** Every convergent sequence is Cauchy.
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Proof. Let \( \{x_n\} \) be a convergent sequence. Let \( x_n \to x \) as \( n \to \infty \). Then for \( \varepsilon > 0 \), there exists an \( n_0 \in \mathbb{N} \) such that

\[
|x_n - x| < \varepsilon/2 \text{ whenever } n \geq n_0.
\]

Hence for \( n, m \geq n_0 \),

\[
|x_n - x_m| \leq |x_n - x| + |x_m - x| < \varepsilon/2 + \varepsilon/2 = \varepsilon.
\]

This implies \( \{x_n\} \) is Cauchy.

\( \square \)

Theorem 2.13. Every Cauchy sequence is bounded.

Proof. Let \( \{x_n\} \) be a Cauchy sequence. We want to show that \( \{x_n\} \) is bounded. Let \( \varepsilon = 1 \). Then there exists an \( n_0 \in \mathbb{N} \) such that

\[
|x_n - x_m| < 1 \text{ whenever } n, m \geq n_0.
\]

In particular,

\[
|x_n - x_{n_0}| < 1 \text{ whenever } n \geq n_0.
\]

This implies

\[
|x_n| \leq 1 + |x_{n_0}| \text{ whenever } n \geq n_0.
\]

Thus

\[
|x_n| \leq M \text{ for all } n \in \mathbb{N}
\]

where

\[
M = \max\{|x_1|, |x_2|, \ldots, |x_{n_0}|, |x_{n_0}| + 1\}.
\]

\( \square \)

Theorem 2.14. A sequence of real numbers \( \{x_n\} \) is Cauchy if and only if \( \{x_n\} \) is convergent.

Proof. Let \( \{x_n\} \) be a Cauchy sequence. We want to show that \( \{x_n\} \) is convergent.

Since \( \{x_n\} \) is Cauchy, it is bounded. By the Bolzano-Weierstrass Theorem, \( \{x_n\} \) has a convergent subsequence \( \{x_{n_k}\} \), say \( x_{n_k} \to a \) as \( k \to \infty \). We want to show that \( x_n \to a \) as \( n \to \infty \).

Let \( \varepsilon > 0 \). Then \( \{x_n\} \) is Cauchy, there exists \( n_1 \in \mathbb{N} \) such that

\[
|x_n - x_m| < \varepsilon/2 \text{ whenever } n, m \geq n_1.
\]

Since \( x_{n_k} \to a \) as \( k \to \infty \), there exists an \( n_2 \in \mathbb{N} \) such that

\[
|x_{n_k} - a| < \varepsilon/2 \text{ whenever } k \geq n_2.
\]

Choose \( k \) so big so that \( n_k \geq n_1 \). Now,

\[
|x_n - a| \leq |x_n - x_{n_k}| + |x_{n_k} - a| < \varepsilon/2 + \varepsilon/2 = \varepsilon
\]

whenever \( n \geq n_1 \). Thus we have \( x_n \to a \) as \( n \to \infty \).

The converse was proven in an earlier theorem.

\( \square \)
Example 2.7. Prove that any real sequence \( \{x_n\} \) that satisfies

\[
|x_n - x_{n+1}| \leq \frac{1}{2^n}, n \in \mathbb{N},
\]
is convergent.

Proof. Let us show that \( \{x_n\} \) is Cauchy. If \( m > n \),

\[
|x_n - x_m| = |x_n - x_{n+1} + x_{n+1} - x_{n+2} + \cdots + x_{m-1} - x_m|
\leq |x_n - x_{n+1}| + |x_{n+1} - x_{n+2}| + \cdots + |x_{m-1} - x_m|
\leq \frac{1}{2^n} + \frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} + \cdots + \frac{1}{2^{m-1}}
= \frac{1}{2^n} \left( 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{m-n}} \right)
= \frac{1}{2^{m-1}} \left( 1 - \frac{1}{2^{m-n}} \right)
< \frac{1}{2^{n-1}}.
\]

Nor for \( \varepsilon > 0 \). Choose \( n_0 \in \mathbb{N} \) so that

\[
\frac{1}{2^{n_0-1}} < \varepsilon
\]
for \( n \geq n_0 \). Thus \( \{x_n\} \) is Cauchy hence convergent.

Remark 2.3. A sequence that satisfies \( x_{n+1} - x_n \to 0 \) is not necessarily Cauchy. For instance, \( \{\log n\} \) satisfies \( |\log(n+1) - \log n| = \log \left( \frac{n+1}{n} \right) \to \log 1 = 0 \) but \( \log n \to \infty \). This shows that \( \{\log n\} \) is not convergent and hence not Cauchy.

2.5 Limits Supremum and Infimum

Consider a real sequence \( \{x_n\} \). Then think of all possible convergent subsequences of \( \{x_n\} \) including the ones converging to \( \pm \infty \). Now the set

\[
L = \{x : x \text{ is a limit of some subsequence of } \{x_n\}\}.
\]

The infimum of the set \( L \) is called the limit inferior of \( \{x_n\} \) and the supremum of \( L \) is called the limit superior of \( \{x_n\} \).

Definition 2.8. Let \( \{x_n\} \) be a real sequence. Then the limit supremum of \( \{x_n\} \) is defined to be

\[
\limsup_{n \to \infty} x_n = \lim_{n \to \infty} \left( \sup_{k \geq n} x_k \right).
\]

Similarly, the limit infimum is defined to be

\[
\liminf_{n \to \infty} x_n = \lim_{n \to \infty} \left( \inf_{k \geq n} x_k \right).
\]
CHAPTER 2. SEQUENCES OF REAL NUMBERS

Note that the lim sup and lim inf of a sequence \( \{ x_n \} \) can very well be the extended real numbers \( \pm \infty \).

**Example 2.8.** Find the lim sup and lim inf of \( \{ \frac{1}{n} \} \).

**Example 2.9.** Find the lim sup and lim inf of \( \{ (-1)^n \} \).

**Theorem 2.15.** Let \( \{ x_n \} \) be a sequence of real numbers. Then

\[
\liminf_{n \to \infty} x_n \leq \limsup_{n \to \infty} x_n.
\]

**Proof.** Since \( \inf_{k \geq n} x_k \leq \sup_{k \geq n} x_k \), we have

\[
\liminf_{n \to \infty} x_k \leq \limsup_{n \to \infty} x_k.
\]

Let \( \{ x_n \} \) be a real sequence and \( \varepsilon > 0 \). By the definition of the limit of a sequence, if \( x_n \to x \), then the interval \( (x - \varepsilon, x + \varepsilon) \) contains all but finitely many terms of the sequence. How does this compare to the lim sup and lim inf of a sequence? If \( t = \liminf x_n \) and \( s = \limsup x_n \), then the interval \( (t - \varepsilon, s + \varepsilon) \) contains all but the finitely many terms of the sequence \( \{ x_n \} \).

We state a few interesting theorems about limit supremum and limit infimum without proof as it might be overwhelming at this level. You can read the proof off the textbook if you are interested.

**Theorem 2.16.**

1. Let \( \{ x_n \} \) be a sequence of real numbers and \( \limsup_{n \to \infty} x_n = s \). Then there exists a subsequence \( \{ x_{n_k} \} \) of \( \{ x_n \} \) such that \( x_{n_k} \to s \) as \( k \to \infty \).

2. Let \( \{ x_n \} \) be a sequence of real numbers and \( \liminf_{n \to \infty} x_n = t \). Then there exists a subsequence \( \{ x_{l_j} \} \) of \( \{ x_n \} \) such that \( x_{l_j} \to t \) as \( j \to \infty \).

**Proof.** Reading Exercise.

**Theorem 2.17.** Let \( \{ x_n \} \) be a sequence of real numbers and \( x \) be an extended real number. Then

\[
\lim_{n \to \infty} x_n = x \text{ if and only if } \liminf_{n \to \infty} x_n = \limsup_{n \to \infty} x_n = x.
\]

**Proof.** Reading Exercise.

**Theorem 2.18.** Let \( \{ x_n \} \) be a sequence and \( x \) be a limit of a subsequence. Then \( \liminf_{n \to \infty} x_n \leq x \leq \limsup_{n \to \infty} x_n \) has a subsequence.

**Theorem 2.19.** If \( x_n \leq y_n \) for \( n \) large, then

\[
\limsup_{n \to \infty} x_n \leq \limsup_{n \to \infty} y_n \text{ and } \liminf_{n \to \infty} x_n \leq \liminf_{n \to \infty} y_n.
\]
Exercise

1. Prove that if a sequence \( \{x_n\} \) is decreasing and bounded below then it converges to a finite limit.

2. Prove that the sequence \( \{2^{1/n}\} \) converges to 1, that is, \( 2^{1/n} \to 1 \) as \( n \to \infty \).
   (Mimic the proof in Example 2.21 Case 2.)

3. Exercise 2.3.2.

4. If \( \{x_n\} \) and \( \{y_n\} \) are Cauchy sequences prove that \( \{x_ny_n\} \) is Cauchy.

5. Find \( \limsup_{n \to \infty} x_n \) and \( \lim_{n \to \infty} x_n \) where \( x_n = \sin(n\pi/2) \).
Chapter 3
Functions on \( \mathbb{R} \)

3.1 Limit of a Function

Definition 3.1. Let \( I \subseteq \mathbb{R} \) be an open interval and \( a \in I \). Let \( f \) be a function defined on \( I \setminus \{a\} \). Then \( f(x) \) is said to have a limit \( L \) as \( x \) approaches to \( a \) if and only if for every \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that

\[
|f(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < |x - \delta| < \varepsilon.
\]

If \( f(x) \) has a limit \( L \) at \( a \) we write

\[
\lim_{x \to a} f(x) = L.
\]

Example 3.1. If \( f(x) = c \) is a constant function defined on \( \mathbb{R} \), then for any \( a \in \mathbb{R} \), \( \lim_{x \to a} f(x) = c \). To see this, for given \( \varepsilon > 0 \) one can choose any \( \delta > 0 \) for which

\[
|f(x) - c| = |c - c| = 0 < \varepsilon
\]

holds automatically for all \( x \) satisfying \( 0 < |x - a| < \delta \).

Example 3.2. For a linear function \( f(x) = mx + c \), where \( m, c \in \mathbb{R} \),

\[
\lim_{x \to a} f(x) = f(a)
\]

for all \( a \in \mathbb{R} \).

Proof. The case \( m = 0 \) is discussed in previous example. So we assume that \( m \neq 0 \). Let \( \varepsilon > 0 \). Choose \( \delta = \varepsilon/m \). Now,

\[
|f(x) - f(a)| = |mx + c - ma - c| = |m||x - a| < |m|\delta = \varepsilon
\]

whenever \( 0 < |x - a| < \delta \).

Example 3.3. Based on what we learned in Calculus I, we know that

\[
\lim_{x \to 2} x^2 = 4.
\]
Let us prove this using our $\varepsilon - \delta$ definition.

Let $\varepsilon > 0$. Here,

$$|f(x) - L| = |x^2 - 4| = |x + 2||x - 2|.$$  

We want to find a $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $0 < |x - 2| < \delta$. If $|x - 2| < 1$ we get $|x| < |x - 2| + 2 < 3$. Hence, $|x + 2| \leq |x| + 2 < 5$. Take $\delta = \min\{1, \varepsilon/5\}$. Then,

$$|x^2 - 4| = |x + 2||x - 2| < 5\delta \leq \varepsilon$$

whenever $0 < |x - 2| < \delta$.

**Theorem 3.1.** Let $I \subseteq \mathbb{R}$ be an open interval and $a \in I$. Let $f, g$ be real functions defined on $I \setminus \{a\}$. Let $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist. Then we have

1. $\lim_{x \to a} (f + g)(x) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$.

2. $\lim_{x \to a} (\alpha f)(x) = \alpha \lim_{x \to a} f(x)$.

3. $\lim_{x \to a} (fg)(x) = \lim_{x \to a} f(x) \lim_{x \to a} g(x)$

4. $\lim_{x \to a} \left( \frac{f}{g} \right)(x) = \lim_{x \to a} \frac{f(x)}{g(x)}$ when $\lim_{x \to a} g(x) \neq 0$.

**Theorem 3.2** (Squeeze Theorem). Let $I \subseteq \mathbb{R}$ be an open interval and $a \in I$. Let $f, g, h$ be real functions defined on $I \setminus \{a\}$. If $f(x) \leq h(x) \leq g(x)$ for all $x \in I \setminus \{a\}$ and

$$\lim_{x \to a} f(x) = L = \lim_{x \to a} g(x),$$

then

$$\lim_{x \to a} h(x) = L.$$

**Theorem 3.3** (Comparison Theorem). Let $I \subseteq \mathbb{R}$ be an open interval and $a \in I$. Let $f, g$ are real functions defined on $I \setminus \{a\}$. If $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist and $f(x) \leq g(x)$ for all $x \in I \setminus \{a\}$, then

$$\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x).$$

**Theorem 3.4.** Let $I \subseteq \mathbb{R}$ be an open interval and $a \in I$. Let $f$ be a real function defined on $I \setminus \{a\}$. Then $\lim_{x \to a} f(x) = L$ exists if and only if $\lim_{n \to \infty} f(x_n) = L$ for every sequence $\{x_n\}, x_n \in I \setminus \{a\}$ which converges to $a$. 
3.2 One-Sided Limits and Limits Involving Infinity

Definition 3.2. Let \( f \) be a real function and \( a, L \in \mathbb{R} \).

1. We say that \( f(x) \) converges to \( L \) as \( x \) approaches \( a \) from left if and only if for given \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that \( f \) is defined on \((a - \delta, a)\) and
   \[
   |f(x) - L| < \varepsilon \text{ whenever } a - \delta < x < a.
   \]
   In this case we write \( \lim_{x \to a^-} f(x) = L \).

2. We say that \( f(x) \) converges to \( L \) as \( x \) approaches \( a \) from right if and only if for given \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that \( f \) is defined on \((a, a + \delta)\) and
   \[
   |f(x) - L| < \varepsilon \text{ whenever } a < x < a + \delta.
   \]
   In this case we write \( \lim_{x \to a^+} f(x) = L \).

3. We say that \( f(x) \) converges to \( L \) as \( x \to \infty \) if and only if for given \( \varepsilon > 0 \) there exists an \( M \in \mathbb{R} \) such that \( f \) is defined on \((M, \infty)\) and
   \[
   |f(x) - L| < \varepsilon \text{ whenever } M < x < \infty.
   \]
   In this case we write \( \lim_{x \to \infty} f(x) = L \).

4. We say that \( f(x) \) converges to \( L \) as \( x \to -\infty \) if and only if for given \( \varepsilon > 0 \) there exists an \( M \in \mathbb{R} \) such that \( f \) is defined on \((-\infty, M)\) and
   \[
   |f(x) - L| < \varepsilon \text{ whenever } -\infty < x < M.
   \]
   In this case we write \( \lim_{x \to -\infty} f(x) = L \).

5. We say that \( f(x) \) converges to \( \infty \) as \( x \to a \) if and only if for given \( M > 0 \) there exists a \( \delta > 0 \) such that \( f \) is defined on \( \{x : 0 < |x - a| < \delta\} \) and
   \[
   f(x) > M \text{ whenever } 0 < |x - a| < \delta.
   \]
   In this case we write \( \lim_{x \to a} f(x) = \infty \).

6. We say that \( f(x) \) converges to \( -\infty \) as \( x \to a \) if and only if for given \( M < 0 \) there exists a \( \delta > 0 \) such that \( f \) is defined on \( \{x : 0 < |x - a| < \delta\} \) and
   \[
   f(x) < M \text{ whenever } 0 < |x - a| < \delta.
   \]
   In this case we write \( \lim_{x \to a} f(x) = -\infty \).
Theorem 3.5. Let \( f \) be a real function. Then

\[
\lim_{x \to a} f(x) = L
\]

if and only if

\[
\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x).
\]

Example 3.4. Let \( f(x) = \begin{cases} x + 1 & \text{if } x \geq 0 \\ x - 1 & \text{if } x < 0 \end{cases} \). Prove that \( \lim_{x \to 0^-} f(x) = 1 \).

Proof. Given \( \varepsilon > 0 \). Take \( \delta = \varepsilon \). Then

\[
|f(x) - 1| = |x| < \varepsilon \text{ whenever } -\delta < x < 0.
\]

Example 3.5. Prove that \( \lim_{x \to 0^+} \sqrt{x} = 0 \).

Proof. Given \( \varepsilon > 0 \). Take \( \delta = \varepsilon^2 \). Then

\[
|f(x) - L| = |\sqrt{x} - 0| = \sqrt{x} < \sqrt{\delta} = \varepsilon \text{ whenever } 0 < x < \delta.
\]

Example 3.6. Prove that \( \lim_{x \to \infty} \frac{1}{x} = 0 \).

Proof. Given \( \varepsilon > 0 \). Take \( M = \frac{1}{\varepsilon} \). Then

\[
|f(x) - L| = \left| \frac{1}{x} - 0 \right| = \frac{1}{|x|} < \frac{1}{M} = \varepsilon
\]

whenever \( M < x < \infty \).

Exercises

1. 3.1.1 (a)
2. 3.1.6 (a)
3. Prove that \( \lim_{x \to 0^+} \frac{1}{x} = \infty \).
3.3 Continuity

In Calculus we learned that a function \( f \) is called continuous at a point \( a \) if

\[
\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a)
\]

which is simply

\[
\lim_{x \to a} f(x) = f(a).
\]

**Definition 3.3.** Let \( E \) be a nonempty subset of \( \mathbb{R} \) and \( f : E \to \mathbb{R} \). Then \( f \) is said to be continuous at a point \( a \in E \) if and only if given \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that

\[
|f(x) - f(a)| < \varepsilon \text{ whenever } |x - a| < \delta \text{ and } x \in E.
\]

The functions \( f \) is said to be continuous on \( E \) if and only if \( f \) is continuous at every point of \( E \).

**Remark 3.1.** Let \( I \subseteq \mathbb{R} \) be an open interval and \( a \in I \). Then a function \( f : I \to \mathbb{R} \) is continuous at \( a \) if and only if

\[
\lim_{x \to a} f(x) = f(a).
\]

**Proof.** \( \lim_{x \to a} f(x) = f(a) \) if and only if given \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that

\[
|f(x) - f(a)| < \varepsilon \text{ whenever } |x - a| < \delta
\]

if and only if \( f \) is continuous at \( a \). Notice that we have to choose \( \delta \) so small that \((a - \delta, a + \delta) \subset I\). Also notice that we did not exclude \( a \) from \((a - \delta, a + \delta)\) because \( f(a) \) is well defined.

**Example 3.7.** The function \( f(x) = \frac{\sin x}{x} \) is not continuous at 0 because the function is not defined at 0.

**Example 3.8.** The function \( f : \mathbb{R} \to \mathbb{R} \) defined by

\[
f(x) = \begin{cases} 
1 & \text{if } x \text{ is rational} \\
-1 & \text{if } x \text{ is irrational}
\end{cases}
\]

is discontinuous everywhere.

**Proof.** Take \( \varepsilon = 1 \). Let \( a \in \mathbb{R} \) be any number. Then for every \( \delta > 0 \), the interval \((a - \delta, a + \delta)\) contains both rational and irrational numbers. If \( a \) is rational, then

\[
|f(x) - f(a)| = |-1 - 1| = 2 > \varepsilon
\]

for all irrational \( x \). Similarly, if \( a \) is irrational, then

\[
|f(x) - f(a)| = |1 + 1| = 2 > \varepsilon
\]

for all rational \( x \). Thus \( f \) is discontinuous at every \( a \in \mathbb{R} \).
Example 3.9. The function $f : \mathbb{Z} \to \mathbb{R}$ defined by $f(x) = x$ is continuous on $\mathbb{Z}$.

**Proof.** Given $\varepsilon > 0$. Let $a \in \mathbb{Z}$ be any point. Choose $0 < \delta < 1$. Then $(a - \delta, a + \delta) = \{a\}$. Then it automatically follows that,

$$|f(x) - f(a)| = |f(a) - f(a)| = 0 < \varepsilon \text{ whenever } |x - a| < \delta.$$

Notice that the only $x$ satisfying $|x - a| < \delta$ is $a$ itself.

**Theorem 3.6.** Suppose that $E$ is a nonempty subset of $\mathbb{R}$, $a \in E$ and $f : E \to \mathbb{R}$. Then the following statements are equivalent.

i. $f$ is continuous at $a \in E$.

ii. If $x_n \in E$ and $x_n \to a$, then $f(x_n) \to f(a)$ as $n \to \infty$.

**Theorem 3.7.** Let $E$ be a nonempty subset of $\mathbb{R}$ and $f, g : E \to \mathbb{R}$. If $f, g$ are continuous at a point $a \in E$, then $f + g, gf$, and $\alpha f, \alpha \in \mathbb{R}$ are continuous. Moreover, $f/g$ is continuous at $a$ when $g(a) \neq 0$.

**Definition 3.4.** Suppose that $A$ and $B$ are subset of $\mathbb{R}$. Let $f : A \to \mathbb{R}$, $g : B \to \mathbb{R}$ and $f(A) \subseteq B$. Then the composition of $g$ with $f$ is the function $g \circ f : A \to \mathbb{R}$ defined by

$$g \circ f(x) = g(f(x)), \quad x \in A.$$

**Theorem 3.8.** Suppose that $A$ and $B$ are subsets of $\mathbb{R}$, that $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ and $f(A) \subseteq B$.

1. If $f$ is continuous at $a \in A$ and $g$ is continuous at $f(a) \in B$, then $g \circ f$ is continuous at $a \in A$.

2. If

   (a) $A = I \setminus \{a\}$, where $I$ is a nondegenerate interval which either contains $a$ or has a one of its endpoints

   (b) $\lim_{x \to a} f(x) = L$ and $L \in B$, and

   (c) $g$ is continuous at $L$,

   then

   $$\lim_{x \to a} g(f(x)) = g\left(\lim_{x \to a} f(x)\right).$$

**Definition 3.5.** Let $E$ be a nonempty subset of $\mathbb{R}$. A function $f : E \to \mathbb{R}$ is said to be bounded on $E$ if and only if there is an $M \in \mathbb{R}$ such that $|f(x)| \leq M$ for all $x \in E$.

**Theorem 3.9** (Extreme Value Theorem). If
(a) $I \subset \mathbb{R}$ is a closed and bounded interval and
(b) $f : I \to \mathbb{R}$ is continuous on $I$, then $f$ is bounded on $I$. Moreover, there exists $x_M, x_m \in I$ such that

$$ f(x_M) = \sup_{x \in I} f(x) \quad \text{and} \quad f(x_m) = \inf_{x \in I} f(x). $$

**Proof.** Let $I \subset \mathbb{R}$ be a closed and bounded interval and $f : I \to \mathbb{R}$ be continuous on $I$. We want to show that $f$ is bounded on $I$.

If possible assume that $f$ is not bounded. Then for every $n \in \mathbb{N}$ there exists $x_n \in I$ such that $|f(x_n)| > n$.

This implies that $|f(x_n)| \to \infty$ as $n \to \infty$.

The sequence $\{x_n\}$ is bounded. By Bolzano–Weierstrass Theorem $\{x_n\}$ has a convergent subsequence, say $x_{n_k} \to a$ as $k \to \infty$. Since $I$ is closed, $a \in I$. This implies that $f(a) \in \mathbb{R}$. So $|f(a)| < \infty$. But

$$ |f(a)| = \lim_{k \to \infty} |f(x_{n_k})| = \infty $$

which is a contradiction. Hence $f$ is bounded.

Let

$$ M = \sup_{x \in I} f(x) \quad \text{and} \quad m = \inf_{x \in I} f(x). $$

We want to show that there is $x_M, x_m \in I$ such that

$$ f(x_M) = M \quad \text{and} \quad f(x_m) = m. $$

Assume on the contrary that $f(x) < M$ for all $x \in I$. Then the function $\frac{1}{M - f(x)}$ is continuous on $I$, hence bounded on $I$. Let

$$ \left| \frac{1}{M - f(x)} \right| \leq C $$

for all $x \in I$. This implies $f(x) \leq M - \frac{1}{C}$ for all $x \in I$. This implies that

$$ M = \sup_{x \in I} f(x) \leq M - \frac{1}{C} $$

which is a contradiction. \hfill \Box

**Theorem 3.10** (Intermediate Value Theorem). Suppose that $a < b$ and $f : [a, b] \to \mathbb{R}$ is continuous. If $y_0$ is a number between $f(a)$ and $f(b)$, then there exists an $x_0 \in (a, b)$ such that $f(x_0) = y_0$. 
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Proof. Without loss of generality, let us assume that $f(a) < f(b)$. Define

$$E = \{x \in [a, b] : f(x) < y_0\}.$$ 

Since $f(a) < y_0 < f(b), a \in E$. So $E$ is nonempty and bounded. This implies that sup $E$ exists, say sup $E = x_0$. Observe that $f(x_0) \leq y_0$ and $x_0 \in [a, b]$ (why?)

We want to show that $f(x_0) = y_0$. If $f(x_0) < y_0$, then the function

$$g(x) = y_0 - f(x)$$

is continuous on $[a, b]$ and $g(x_0) = y_0 - f(x_0) > 0$. Let $\varepsilon = g(x_0)/2$. Since $g$ is continuous at $x_0$, there exists $\delta > 0$ such that $\{x : |x - x_0| < \delta\} \subset [a, b]$ and

$$|g(x) - g(x_0)| < \frac{g(x_0)}{2}$$

whenever $|x - x_0| < \delta$. This implies

$$\frac{g(x_0)}{2} < g(x)$$

whenever $|x - x_0| < \delta$. Then for any $x_1 \in (x_0, x_0 + \delta),$

$$0 < \frac{g(x_0)}{2} < g(x_1) = y_0 - f(x_1) \implies f(x_1) < y_0.$$ 

Thus we have $x_1 > x_0$ and $x_1 \in E$. This implies that $x_0 \neq \sup E$, a contradiction. Hence $f(x_0) = y_0$. Also it follows that $x_0 \neq a$ and hence $x_0 \in (a, b)$.

$\square$